

On a class of hereditary crossed-product orders*

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Abstract

In this brief note, we revisit a class of crossed-product orders over discrete valuation rings introduced by D. E. Haile. We give simple but useful criteria, which involve only the two-cocycle associated with a given crossed-product order, for determining whether such an order is a hereditary order or a maximal order.

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If R is a ring, then $J(R)$ will denote its Jacobson radical, $U(R)$ its group of multiplicative units, and $R^\#$ the subset of all the non-zero elements. The terminology used in this paper, if not in [1], can be found in [3]. The book by Reiner [3] is also an excellent source of literature on maximal orders and hereditary orders.

Let V be a discrete valuation ring (DVR), with quotient field F , and let K/F be a finite Galois extension, with group G , and let S be the integral closure of V in K . Let $f \in Z^2(G, U(K))$ be a normalized two-cocycle. If $f(G \times G) \subseteq S^\#$, then one can construct a “crossed-product” V -algebra

$$A_f = \sum_{\sigma \in G} Sx_\sigma,$$

with the usual rules of multiplication ($x_\sigma s = \sigma(s)x_\sigma$ for all $s \in S, \sigma \in G$ and $x_\sigma x_\tau = f(\sigma, \tau)x_{\sigma\tau}$). Then A_f is associative, with identity $1 = x_1$, and center $V = Vx_1$. Further, A_f is a V -order in the crossed-product F -algebra $\Sigma_f = \sum_{\sigma \in G} Kx_\sigma = (K/F, G, f)$.

Two such cocycles f and g are said to be cohomologous over S (respectively cohomologous over K), denoted by $f \sim_S g$ (respectively $f \sim_K g$), if there are elements $\{c_\sigma \mid \sigma \in G\} \subseteq U(S)$ (respectively $\{c_\sigma \mid \sigma \in G\} \subseteq K^\#$) such that $g(\sigma, \tau) = c_\sigma \sigma(c_\tau) c_{\sigma\tau}^{-1} f(\sigma, \tau)$ for all $\sigma, \tau \in G$. Following [1], let $H = \{\sigma \in G \mid f(\sigma, \sigma^{-1}) \in U(S)\}$. Then H is a subgroup of G . On G/H , the left coset space of G by H , one can define a partial ordering by the rule $\sigma H \leq \tau H$ if $f(\sigma, \sigma^{-1}\tau) \in U(S)$. Then “ \leq ” is well-defined and depends only on the cohomology class of f over S . Further, H is the unique least element. We call this partial ordering on G/H the *graph of f* .

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Such a setup was first formulated by Haile in [1], with the assumption that S is unramified over V , wherein, among other things, conditions equivalent to such orders being maximal orders were considered. This is the class of crossed-product orders we shall study in this paper, *always assuming that S is unramified over V* . We emphasize the fact that, since we do not require that $f(G \times G) \subseteq U(S)$, this theory constitutes a drastic departure from the classical theory of crossed-product orders over DVRs, such as can be found in [2].

Let us now fix additional notation to be used in the rest of the paper, most of it borrowed from [1] as before. If M is a maximal ideal of S , let D_M be the decomposition group of M , let K_M be the decomposition field, and let S_M be the localization of S at M . The two-cocycle $f : G \times G \mapsto S^\#$ yields a two-cocycle $f_M : D_M \times D_M \mapsto S_M^\#$, determined by the restriction of f to $D_M \times D_M$ and the inclusion of $S^\#$ in $S_M^\#$. Then $A_{f_M} = \sum_{\sigma \in D_M} S_M x_\sigma$ is a crossed-product order in $\Sigma_{f_M} = \sum_{\sigma \in D_M} K x_\sigma = (K/K_M, D_M, f_M)$. In addition, we can obtain a *twist* of f , described in [1, pp. 137-138] and denoted by \tilde{f} , which depends on the choice of a maximal ideal M of S , and the choice of a set of coset representatives of D_M in G . We also define $F : G \times G \mapsto S^\#$ by $F(\sigma, \tau) = f(\sigma, \sigma^{-1}\tau)$ for $\sigma, \tau \in G$. While \tilde{f} is a two-cocycle, F is not.

If B is a V -order of Σ_f containing A_f , then by [1, Proposition 1.3], $B = A_g = \sum_{\sigma \in G} S y_\sigma$ for some two-cocycle $g : G \times G \mapsto S^\#$, with $g \sim_K f$. Moreover, the proof of [1, Proposition 1.3] shows that $y_\sigma = k_\sigma x_\sigma$ for some $k_\sigma \in K^\#$, with $k_1 = 1$, whence g is also a normalized two-cocycle.

We begin with a technical result.

Sublemma. *Let $\tau \in G$. If $I_\tau = \prod_{f(\tau, \tau^{-1}) \notin M} M$, where M denotes a maximal ideal of*

S , then $I_\tau^{\tau^{-1}} = I_{\tau^{-1}}$.

Proof. We have

$$I_\tau^{\tau^{-1}} = \prod_{f(\tau, \tau^{-1}) \notin M} M^{\tau^{-1}} = \prod_{f^{\tau^{-1}}(\tau, \tau^{-1}) \notin M^{\tau^{-1}}} M^{\tau^{-1}} = \prod_{f(\tau^{-1}, \tau) \notin M^{\tau^{-1}}} M^{\tau^{-1}} = I_{\tau^{-1}}.$$

□

Theorem. *The crossed-product order A_f is hereditary if and only if $f(\tau, \tau^{-1}) \notin M^2$ for all $\tau \in G$ and every maximal ideal M of S .*

Proof. The theorem obviously holds if $H = G$, in which case A_f is an Azumaya algebra over V , so let us assume from now on that $H \neq G$.

Suppose A_f is hereditary. First, assume A_f is a maximal order and S is a DVR. Let v be the valuation corresponding to S with value group \mathbb{Z} . Then by [1, Theorem 2.3], H is a normal subgroup of G and G/H is cyclic. Further, there exists $\sigma \in G$ such that $v(f(\sigma, \sigma^{-1})) \leq 1$, $G/H = \langle \sigma H \rangle$, and the graph of f is the chain $H \leq \sigma H \leq \sigma^2 H \leq \dots \leq \sigma^{m-1} H$, where $m = |G/H|$. Choose j maximal such that $1 \leq j \leq m-1$ and $v(f(\sigma^j, \sigma^{-j})) \leq 1 \forall 1 \leq i \leq j$. We always have $\sigma H \leq \sigma^{-j} H$; but if $j < m-1$, then we also have $\sigma^j H \leq \sigma^{j+1} H$.

Hence if $j < m - 1$, then, from the cocycle identity $f^{\sigma^j}(\sigma, \sigma^{-j}\sigma^{-1})f(\sigma^j, \sigma^{-j}) = f(\sigma^j, \sigma)f(\sigma^{j+1}, \sigma^{-j}\sigma^{-1})$, we conclude that $v(f(\sigma^i, \sigma^{-i})) \leq 1 \forall 1 \leq i \leq j + 1$, a contradiction. So we must have $j = m - 1$, so that $v(f(\sigma^i, \sigma^{-i})) \leq 1 \forall 1 \leq i \leq m - 1$. If τ is an arbitrary element of G , then $\tau = \sigma^i h$ for some $h \in H$ and some integer i , $0 \leq i \leq m - 1$. Therefore, by [1, Lemma 3.6], $v(f(\tau, \tau^{-1})) = v(F(\sigma^i h, 1)) = v(F(\sigma^i, 1)) = v(f(\sigma^i, \sigma^{-i})) \leq 1$; that is, $f(\tau, \tau^{-1}) \notin J(S)^2$.

We maintain the assumption that A_f is a maximal order, but we now drop the condition that S is a DVR. By [1, Theorem 3.16], there exists a twist of f , say \tilde{f} , such that $f \sim_S \tilde{f}$. By [1, Corollary 3.11], for any maximal ideal M of S , A_{f_M} is a maximal order in Σ_{f_M} ; hence $f_M(\tau, \tau^{-1}) \notin M^2 \forall \tau \in D_M$ by the preceding paragraph. Therefore, from the manner in which \tilde{f} is constructed from f , we infer that $\tilde{f}(\tau, \tau^{-1}) \notin M^2 \forall \tau \in G$ and any maximal ideal M of S , and thus $f(\tau, \tau^{-1}) \notin M^2 \forall \tau \in G$ and every maximal ideal M of S , since $f \sim_S \tilde{f}$.

If A_f is not a maximal order, then it is the intersection of finitely many maximal orders, say $A_{f_1}, A_{f_2}, \dots, A_{f_l}$. Note that

$$A_{f_i} = \sum_{\tau \in G} S y_{\tau}^{(i)} = \sum_{\tau \in G} S k_{\tau}^{(i)} x_{\tau},$$

for some $k_{\tau}^{(i)} \in K$. Fix a $\sigma \in G$, and a maximal ideal N of S . Let v_N be the valuation corresponding to N , with value group \mathbb{Z} . Since

$$S = \bigcap_{i=1}^l S k_{\sigma}^{(i)},$$

there exists i_0 such that $v_N(k_{\sigma}^{(i_0)}) = 0$. Let $g = f_{i_0}$ and, for $\tau \in G$, let $k_{\tau} = k_{\tau}^{(i_0)}$ and $y_{\tau} = y_{\tau}^{(i_0)}$, so that $A_g = \sum_{\tau \in G} S k_{\tau} x_{\tau} = \sum_{\tau \in G} S y_{\tau}$. By [1, Proposition 3.1], $J(A_f) = \sum_{\tau \in G} I_{\tau} x_{\tau}$ and $J(A_g) = \sum_{\tau \in G} J_{\tau} y_{\tau}$, where

$$I_{\tau} = \prod_{f(\tau, \tau^{-1}) \notin M} M \quad \text{and} \quad J_{\tau} = \prod_{g(\tau, \tau^{-1}) \notin M} M,$$

and M denotes a maximal ideal of S . Since A_f is a hereditary V -order in Σ_f and $A_f \subseteq A_g \subseteq \Sigma_f$, we have $J(A_g) \subseteq J(A_f)$, from which we conclude that $J_{\sigma^{-1}} y_{\sigma^{-1}} \subseteq I_{\sigma^{-1}} x_{\sigma^{-1}}$ and so $J_{\sigma^{-1}} k_{\sigma^{-1}} \subseteq I_{\sigma^{-1}}$. We have $y_{\sigma^{-1}} J_{\sigma} y_{\sigma} = k_{\sigma^{-1}} x_{\sigma^{-1}} J_{\sigma} k_{\sigma} x_{\sigma} = J_{\sigma^{-1}} k_{\sigma^{-1}} x_{\sigma^{-1}} k_{\sigma} x_{\sigma} \subseteq I_{\sigma^{-1}} x_{\sigma^{-1}} k_{\sigma} x_{\sigma} = \sigma^{-1}(k_{\sigma}) I_{\sigma^{-1}} f(\sigma^{-1}, \sigma) = (k_{\sigma} I_{\sigma} f(\sigma, \sigma^{-1}))^{\sigma^{-1}}$. On the other hand, $y_{\sigma^{-1}} J_{\sigma} y_{\sigma} = J_{\sigma}^{\sigma^{-1}} g(\sigma^{-1}, \sigma) = J_{\sigma^{-1}} g(\sigma^{-1}, \sigma)$. Since A_g is a maximal order and therefore $g(\sigma^{-1}, \sigma) \notin M^2$ for every maximal ideal M of S , we see that $J_{\sigma^{-1}} g(\sigma^{-1}, \sigma) = J(V)S$ and so $y_{\sigma^{-1}} J_{\sigma} y_{\sigma} = J(V)S$. Therefore $J(V)S \subseteq k_{\sigma} I_{\sigma} f(\sigma, \sigma^{-1})$. Since $v_N(k_{\sigma}) = 0$, we conclude that $f(\sigma, \sigma^{-1}) \notin N^2$, and so $f(\tau, \tau^{-1}) \notin M^2 \forall \tau \in G$ and any maximal ideal M of S .

Conversely, suppose that $f(\tau, \tau^{-1}) \notin M^2$ for every maximal ideal M of S and every $\tau \in G$. Let $B = O_l(J(A_f))$, the left order of $J(A_f)$; that is,

$B = \{x \in \Sigma_f \mid xJ(A_f) \subseteq J(A_f)\}$. Since $\Sigma_f \supseteq B \supseteq A_f$, $B = \sum_{\tau \in G} Sk_\tau x_\tau$, for some $k_\tau \in K^\#$. For each $\tau \in G$, we have $S \subseteq Sk_\tau$, and we will now show that $S = Sk_\tau$. As above, write $J(A_f) = \sum I_\tau x_\tau$, with $I_\tau = \prod M$, where the product is taken over all maximal ideals M of S for which $f(\tau, \tau^{-1}) \notin M$. Observe that $J(V)S = I_1 \supseteq k_\tau x_\tau I_{\tau^{-1}} x_{\tau^{-1}} = k_\tau I_{\tau^{-1}}^\tau f(\tau, \tau^{-1}) = k_\tau I_\tau f(\tau, \tau^{-1})$. Since $f(\tau, \tau^{-1}) \notin M^2$ for every maximal ideal M of S , we must have $I_\tau f(\tau, \tau^{-1}) = J(V)S$, and so $J(V)S \supseteq k_\tau J(V)S \supseteq J(V)S$ and thus $S = Sk_\tau$, as desired. This shows that $O_l(J(A_f)) = A_f$ and A_f is hereditary. \square

Not only can this criterion enable one to rapidly determine whether or not the crossed-product order A_f is hereditary, the utility of the theorem above is now demonstrated by the ease with which the following corollaries of it are obtained.

Corollary 1. *The crossed-product order A_f is hereditary if and only if $f(\tau, \gamma) \notin M^2$ for all $\tau, \gamma \in G$ and every maximal ideal M of S .*

Proof. This follows from the cocycle identity $f^\tau(\tau^{-1}, \tau\gamma)f(\tau, \gamma) = f(\tau, \tau^{-1})$. \square

In other words, the order A_f is hereditary if and only if the values of the two-cocycle f are all square-free.

Since A_f is a maximal order if and only if it is hereditary and primary, by combining our result and results in [1], we immediately have the following.

Corollary 2. *Given a crossed-product order A_f ,*

1. *it is a maximal order if and only if for every maximal ideal M of S , $f(\tau, \tau^{-1}) \notin M^2$ for all $\tau \in G$, and there exists a set of right coset representatives g_1, g_2, \dots, g_r of D_M in G (i.e., G is the disjoint union $\cup_i D_M g_i$) such that for all i , $f(g_i, g_i^{-1}) \notin M$.*
2. *if S is a DVR, then it is a maximal order if and only if $f(\tau, \tau^{-1}) \notin J(S)^2$ for all $\tau \in G$.*

Proof. In either case, the primarity of A_f is guaranteed by [1, Theorem 3.2] (see also [1, Proposition 2.1(b)] when S is a DVR). \square

The Theorem above can readily be put to effective use with the crossed-product orders in [1, §4], for example. In that section, all the crossed-product orders involved are primary orders, and the two-cocycles are given in tabular form, with the values factorized into primes of S . Using our criterion, it now becomes a straightforward process to determine which of those orders are maximal orders and which are not, by simply consulting, in each case, the given table of values for the two-cocycle; the table whose entries are all square-free represents a maximal order. This determination can be made with little effort! In fact, if one knows that the crossed-product order A_f is a primary order, then determining whether or not it is a maximal order could even be easier, as the following result shows.

Corollary 3. *Suppose the crossed-product order A_f is primary. Then it is a maximal order if and only if there exists a maximal ideal M of S such that $f(\tau, \tau^{-1}) \notin M^2$ for all $\tau \in D_M$.*

Proof. This follows from [1, Corollary 3.11 and Proposition 2.1(b)]. \square

Let L be an intermediate field of F and K , let G_L be the Galois group of K over L , let U be a valuation ring of L lying over V , and let T be the integral closure of U in K . Then one can obtain a two-cocycle $f_{L,U} : G_L \times G_L \mapsto T^\#$ from f by restricting f to $G_L \times G_L$ and embedding $S^\#$ in $T^\#$. As before, $A_{f_{L,U}} = \sum_{\tau \in G_L} T x_\tau$ is a U -order in $\Sigma_{f_{L,U}} = \sum_{\tau \in G_L} K x_\tau = (K/L, G_L, f_{L,U})$.

Corollary 4. *Suppose the crossed-product order A_f is hereditary. Then $A_{f_{L,U}}$ is a hereditary order in $\Sigma_{f_{L,U}}$ for each intermediate field L of F and K and for every valuation ring U of L lying over V .*

This leads to the following.

Corollary 5. *Suppose the crossed-product order A_f is hereditary. Then A_{f_M} is a maximal order in Σ_{f_M} for each maximal ideal M of S .*

Proof. The order A_{f_M} is always primary, by [1, Proposition 2.1(b)]. \square

The following example illustrates two limitations of our theory, however.

Example. We give two crossed-product orders A_{f_1} and A_{f_2} with $f_1 \sim_K f_2$ and the graphs of f_1 and f_2 identical, but A_{f_1} is hereditary while A_{f_2} is not. Also, we give an example to demonstrate that the converse of Corollary 5 does not always hold.

Let $F = \mathbb{Q}(x)$, and let $K = \mathbb{Q}(i)(x)$. Then the Galois group $G = \langle \sigma \rangle$ is a group of order two, where σ is induced by the complex conjugation on $\mathbb{Q}(i)$. If $V = \mathbb{Q}[x]_{(x^2+1)}$, then S has two maximal ideals, namely $M_1 = (x+i)S$ and $M_2 = (x-i)S$, and $D_{M_1} = D_{M_2} = \{1\}$. Let $f_1, f_2 : G \times G \mapsto S^\#$ be two-cocycles defined by $f_j(1, 1) = f_j(1, \sigma) = f_j(\sigma, 1) = 1$ and $f_1(\sigma, \sigma) = (x^2 + 1)x$, $f_2(\sigma, \sigma) = (x^2 + 1)^2x$.

Then $f_1 \sim_K f_2$, and the subgroup of G associated with either cocycle is $H = \{1\}$, so that the graphs of f_1 and f_2 are identical. Clearly, A_{f_1} is hereditary but A_{f_2} is not. We conclude that the property that a crossed-product order A_f is hereditary is not an intrinsic property of the graph of f .

Also, if we set $f = f_2$, we see that $A_{f_M} = S_M$ for each maximal ideal M of S , and therefore A_{f_M} is a maximal order in $\Sigma_{f_M} = K$ for each maximal ideal M of S , and yet A_f is not even hereditary (cf. [1, Corollary 3.11], and [2, Theorem 1]). This is the case because A_f is not primary, and also because $f(G \times G) \not\subseteq U(S)$. \square

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